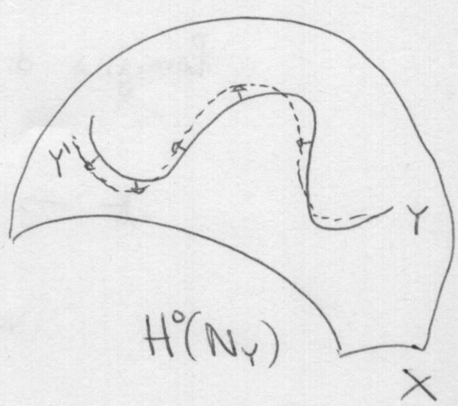
TEORIA DELLE DEFORMAZIONI

§1. IL FIBRATO (CO)NORMALE

$$Y \subset X \qquad 0 \to T_Y \to (T_X)_{|Y} \to N_{Y/X} \to 0$$



FATTO $\quad N_{Y/X} \simeq \left(I/I^2\right)^\vee, \quad \left(I = I_Y \subset \mathcal{O}_X\right)$

casi particolari:

ⓐ codim $Y = 1 \qquad I_Y = \mathcal{O}_X(-Y)$

$$\Rightarrow N_Y = \left(I/I^2\right)^\vee \overset{\star}{=} \left(\mathcal{O}_X(-Y)_{|Y}\right)^\vee = \mathcal{O}_Y(Y)$$

$$\left[\begin{array}{l} \star \quad \underset{\text{ideale}}{I \subset A} \rightsquigarrow A/I \otimes I \simeq I/I^2 \\ \quad \Rightarrow (I_Y)_{|Y} := I_Y \otimes \mathcal{O}_Y = I_Y \otimes \dfrac{\mathcal{O}_X}{I_Y} \simeq \dfrac{I_Y}{I_Y^2} \end{array}\right]$$

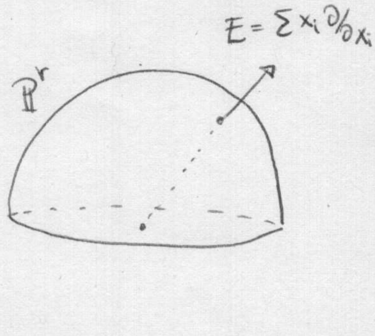
$\Rightarrow$ Se $S = X$ superficie, $\quad \deg(N_{C/S}) = C^2$

ⓑ $C \subset \mathbb{P}^r \qquad d = \deg C, \quad g = g(C)$

$$\Rightarrow \chi(N_C) = (r+1)d + (r-3)(1-g)$$

prova: $\quad \chi(N_C) = \chi(T_{\mathbb{P}|C}) - \chi(T_C)$

$$\chi(T_C) \overset{RR}{=} \underset{\substack{\| \\ 1-g}}{\chi(\mathcal{O}_C)} + \underset{\substack{\| \\ 2-2g}}{\deg T_C} = 3-3g$$



EULERO:

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus r+1} \to T_{\mathbb{P}^r} \to 0$$

$$\underset{\text{a } C}{\overset{\text{restr}}{\rightsquigarrow}} \quad \chi(T_{\mathbb{P}^r|C}) = \chi\left(\mathcal{O}_{\mathbb{P}^r}(1)^{\oplus r+1}\right) - \chi(\mathcal{O}_{\mathbb{P}^r})$$

$$= (r+1)(1-g+d) - 1 + g \qquad \square$$

§2. LO SCHEMA DI HILBERT

famiglia di sottoschemi (chiusi) di $\mathbb{P}^r$? $\{X_t\}_{t\in B}$

$\mathcal{X} \to B$ t.c. $\mathcal{X}_t = X_t$ (fibra su $t \in B$)

voglio $\mathcal{X} \subset \mathbb{P}^r \times B$

fisso $p = p(m) \in \mathbb{Q}[m]$.

$$\mathcal{H}_p := \{ [X] : X \subset \mathbb{P}^r , \ p_X = p \}$$

Thm

ⓐ $\exists\, \mathcal{H}_p$ (schema proiett.) ed esiste

$\mathcal{X} \subset \mathbb{P}^r \times \mathcal{H}_p$ "famiglia universale";

i.e. $\forall$ altra famiglia $\mathcal{X}' \to B$ si ha:

(di sottosch. di $\mathbb{P}^r$ con $p_X = p$)

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ B \arrow[r, dashed, "\exists!"] & \mathcal{H}_p \end{tikzcd}$$

i.e. $\mathcal{X}' = B \times_{\mathcal{H}_p} \mathcal{X}$

ⓑ $T_{[X]}\mathcal{H}_p \simeq H^0(N_{X/\mathbb{P}^r})$

se X é loc. int. compl.:

ⓒ se $h^1(N_X) = 0 \Rightarrow \mathcal{H}_p$ non sing. in $[X]$

ⓓ $h^0(N_X) - h^1(N_X) \leq \dim_{[X]} \mathcal{H}_p \leq h^0(N_X)$

§3. ESEMPI

① CURVE PIANE $\quad C : \{ f(x,y,z) = 0 \} \subset \mathbb{P}^2, \quad H^0(\mathcal{O}(1)) := \langle x, y, z \rangle$

$$\mathcal{X}_d = \{ (\vec{x}, f) \in \mathbb{P}^2 \times \underbrace{\mathbb{P} H^0(\mathcal{O}(d))}_{\mathbb{P}^{N_d}} : f(\vec{x}) = 0 \}$$

$$\mathcal{X}_d \longrightarrow \mathbb{P}^{N_d}$$

$N_d = \binom{d+2}{2} - 1, \qquad \chi(N_{C/\mathbb{P}^2}) = g - 1 + 3d$

$g = g(C) = \binom{d-1}{2} \qquad\qquad = N_d$

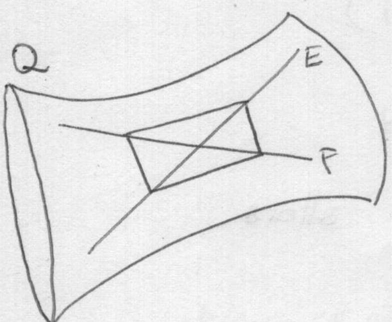
Claim: $h^1(\mathcal{N}_C) = 0$: $\qquad \omega_C = \omega_{\mathbb{P}^2} \otimes \mathcal{O}_C(C)$

$$\Rightarrow \quad d^2 = C^2 = \deg \mathcal{N}_C > \deg K_C = d^2 - 3d. \quad \square$$

$$\boxed{\chi(\mathcal{N}_C) = N_d}$$

② CURVE SULLE QUADRICHE $\quad \mathcal{C}_{a,b} = \{ [C] :$ C curva giacente su qualche quadrica $Q \subset \mathbb{P}^3$, bideg $C = (a,b) \}$

digressione: geometria della quadrica $Q = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$



$Q \approx S^2 \times S^2 \Rightarrow b_1 = 0, b_3 = 0, \quad b_2 = 2$

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

$K_Q \sim (K_{\mathbb{P}^3} + 2H_Q)|_Q < 0 \Rightarrow h^{2,0} = h^0(K) = 0$

succ. esp. $\Rightarrow \quad \mathrm{PIC} Q \simeq H^2(Q, \mathbb{Z})$

$\simeq \mathbb{Z} \cdot E \oplus \mathbb{Z} \cdot F \qquad E = \{0\} \times \mathbb{P}^1$, $F = \mathbb{P}^1 \times \{0\}$

(scelgo il piano tg)

$H_Q \sim E + F \qquad E^2 = 0, \; F^2 = 0$

$K_Q \sim -2E - 2F \qquad E \cdot F = 1$

$C \subset Q, \quad C \sim aE + bF \qquad a, b > 0$

$C^2 = 2ab, \quad \deg_{\mathbb{P}^3}(C) = C \cdot H_Q = a + b$

$2g - 2 = C^2 + C \cdot K_Q \rightsquigarrow \quad g(C) = (a-1)(b-1)$

KÜNNETH: $\quad H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a,b)) = H^0(\mathbb{P}^1, \mathcal{O}(a)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(b))$

$H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a,b)) = H^0(\mathcal{O}_{\mathbb{P}^1}(a)) \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(b)) \oplus H^1(\mathcal{O}(a)) \otimes H^0(b)$

$$\Rightarrow \quad \dim |C| = h^0(\mathcal{O}_Q(a,b)) - 1 = ab + a + b$$

Siano ora: $\mathcal{H} := \mathrm{Hilb}^3_{d,g}$ $\quad d = a+b$, $\quad g = (a-1)(b-1)$

$\mathbb{P}^9 = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ (spazio delle quadriche)

$$W := \{ (Q,C) \in \mathbb{P}^9 \times \mathcal{H} : \ C \in |\mathcal{O}_Q(a,b)| \}$$

(meglio: lavoriamo in realtà sugli aperti $\mathcal{U} \subset \mathbb{P}^9$, $V \subset \mathcal{H}$ che evitano oggetti singolari!)

$W \downarrow \mathbb{P}^9$ dominante, fibre $\simeq |C|$

$$\Rightarrow \dim W = 9 + \dim|C| = 9 + ab + a + b$$

Sia $\varphi : W \to \mathcal{H}$ la seconda proiezione.

Rmk $\overline{\mathcal{C}_{a,b}} = \overline{\varphi(W)}$; le fibre: se $C \in \varphi(W)$,

$$\varphi^{-1}(C) = \{\text{quadriche contenenti } C\} = \mathbb{P}(H^0(\mathcal{I}_C(2)))$$

$\Rightarrow$ C contenuta in 1 o infinite quadriche (in nessuna quadrica se $C \notin \varphi(W)$)

ma se $C \subset Q_1 \cap Q_2 \Rightarrow \deg C \leq 4$

Conseguenza: se ~~deg~~ $a+b \geq 5$ allora

$$\dim \mathcal{C}_{a,b} = \dim W = 9 + ab + a + b$$

CALCOLO DEL FIBRATO NORMALE

Vogliamo capire $\mathcal{N}_C = \mathcal{N}_{C/\mathbb{P}^3}$ per $C \in \mathcal{C}_{a,b}$

Sia $C \subset Q$. $\omega_C = \omega_Q \otimes N_{C/Q}$, da cui

$\deg \mathcal{N}_{C/Q} > \deg \omega_C$, quindi $h^1(\mathcal{N}_{C/Q}) = 0$ e

$$h^0(\mathcal{N}_{C/Q}) \overset{RR}{=} \dim |C| = ab+a+b$$

Considero $\mathcal{N}_Q := N_{Q/\mathbb{P}^3} \simeq \mathcal{O}_Q(2)$

$$(\mathcal{N}_Q)_{|C} \simeq \mathcal{O}_C(2)$$

Twistando la sequenza esatta corta di $C \subset Q$ per $\mathcal{O}_Q(2)$ ottengo:

$$0 \to \underset{\mathcal{O}_Q(2-a,2-b)}{\underset{\parallel}{\mathcal{O}_Q(2-C)}} \to \underset{\mathcal{N}_Q}{\underset{\parallel}{\mathcal{O}_Q(2)}} \to \underset{(\mathcal{N}_Q)_{|C}}{\underset{\parallel}{\mathcal{O}_C(2)}} \to 0$$

da cui: $h^0(\mathcal{N}_Q|_C) = g - h^0(\mathcal{O}_Q(2-a, 2-b))$.

Considerando la sequenza esatta

$$0 \to \mathcal{N}_{C/Q} \to \mathcal{N}_C \to (\mathcal{N}_Q)_{|C} \to 0$$

si ottiene $h^0(\mathcal{N}_C) = ab+a+b+g-h^0(\mathcal{O}_Q(2-a,2-b))$

In particolare: <u>se $a,b \geq 3$</u>, $h^0(\mathcal{N}_C) = ab+a+b+g \underset{\underset{(a+b\geq 5)}{\uparrow\text{se}}}{=} \dim \mathcal{C}_{a,b}$

[lo stesso vale per $(a,b)=(2,3)$] $\rightsquigarrow$ $\Longrightarrow$ $\mathcal{C}_{a,b}$ è un aperto di una componente irriducibile non-singolare in ogni punto $[C] \in \mathcal{C}_{a,b}$ di $\mathcal{H} = \mathrm{Hilb}^3_{d,g}$)

Rimangono da studiare i casi $(a,b) = (1,1), (1,2), (1,3), (2,2)$ in cui $\mathcal{C}_{a,b}$ ha dimensione più piccola (va calcolato $h^0(\mathcal{I}_{C/\mathbb{P}^3}(2))$)

Osservazione se $a,b \geq 4$ si trova $h^1(\mathcal{N}_C) \neq 0$ (ma $\mathcal{H}$ é cmq liscio!)

[usare $h^1(\mathcal{N}_C) = h^1(\mathcal{N}_Q|_C) = h^1(\mathcal{O}_C(2))$]

3 CURVE INTERSEZIONI COMPLETE IN $\mathbb{P}^3$

$A \in |\mathcal{O}_{\mathbb{P}^3}(a)|$, $B \in |\mathcal{O}_{\mathbb{P}^3}(b)|$ generiche, $a, b \geq 1$

$C = A \cap B$, $\deg C = ab$

$N_A = \mathcal{O}_A(a)$, $N_B = \mathcal{O}_B(b)$

Claim: $\mathcal{N}_C \simeq (N_A \oplus N_B)_{|C} = \mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$

idea 1: $\mathcal{N}_C = \dfrac{T_{\mathbb{P}}}{T_C} \xrightarrow{\varphi} \dfrac{T_{\mathbb{P}}}{T_A} \oplus \dfrac{T_{\mathbb{P}}}{T_B} = N_A \oplus N_B$ (mappa quoziente)

φ iniettiva: $T_C = T_A \cap T_B$

$\Longrightarrow$ anche suriettiva, poichè fibrati dello stesso rango. □

idea 2: risolvo $\mathcal{I}_C \subset \mathcal{O}_{\mathbb{P}}$, $\mathcal{I}_C = (f, g)$ $\quad A : f = 0$, $B : g = 0$

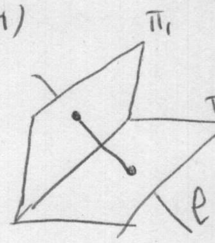
$$0 \to \mathcal{O}_{\mathbb{P}}(-a-b) \xrightarrow{(-g, f)} \mathcal{O}_{\mathbb{P}}(-a) \oplus \mathcal{O}_{\mathbb{P}}(-b) \to \mathcal{I}_C \to 0$$

$$0 \to \mathcal{H}om(\mathcal{I}_C, \mathcal{O}_C) \xrightarrow{\simeq} \mathcal{O}_C(a) \oplus \mathcal{O}_C(b) \xrightarrow{0} \mathcal{O}_C(a+b)$$ (perchè $f = g = 0$ su C)

$\left[\text{ma } \mathcal{N}_C = \mathcal{H}om_{\mathcal{O}_C}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}, \mathcal{O}_C) \right]$ □

Esempio: se $C = \ell$ retta di $\mathbb{P}^3$, $N_\ell = \mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(1)$

$$\Longrightarrow h^0(\mathcal{N}_\ell) = 4 = \dim \mathbb{G}(1,3)$$



IL CANONICO: $\omega_C = \omega_{\mathbb{P}^3} \otimes \det \mathcal{N}_C$
$= \mathcal{O}_C(a+b-4)$

$\Longrightarrow \deg \omega_C = \deg C \cdot (a+b-4) = ab(a+b-4)$

da cui: $g(C) = \frac{1}{2} ab(a+b-4) + 1$.

④ GRASSMANNIANE

$$G(k,n) := \{ k\text{-piani di } \mathbb{C}^n \}$$

$$\mathbb{G}(k,n) := \{ \text{sottospazi } \mathbb{P}^k \subset \mathbb{P}^n \} \rightsquigarrow$$ Schema di Hilbert dei sottospazi lineari di dimensione k in $\mathbb{P}^n$

$$\simeq G(k+1, n+1)$$

$$P : G(k,n) \hookrightarrow \mathbb{P}\Lambda^k \mathbb{C}^n$$ (Plücker Embedding)

$$V = \langle v_1, \dots, v_k \rangle \mapsto v_1 \wedge \dots \wedge v_k$$

Esplicitamente (fisso una base di $\mathbb{C}^n$):

$$V \rightsquigarrow (a_{ij}) \in \mathbb{C}^{k \times n}$$ (Vedo V come spazio generato dalle righe di una matrice)

$$\rightsquigarrow P(V) = (\dots : p_{i_1 \cdots i_k} : \dots),$$ $p_{i_1 \cdots i_k}$ = minore di (a_{ij}) ottenuto cancellando le colonne $i_1, \dots, i_k$

(la matrice (a_{ij}) associata a V non è unica, ma i minori mod. ~~scalar~~ multipli sì)

Sia $\Gamma = \mathbb{C}^{n-k} \subset \mathbb{C}^n$ fissato.

$$\mathcal{U}_\Gamma := \{ V \in G(k,n) : V \cap \Gamma = \{0\} \}$$ (aperto)

Proposizione $\mathcal{U}_\Gamma \simeq \mathbb{A}^{k(n-k)}$

idea: se, e.g., $\Gamma = \{ x_1 = 0, \dots, x_k = 0 \}$ $\rightsquigarrow V \oplus \Gamma = \mathbb{C}^n$

$$\mathcal{U}_\Gamma = \{ V : \text{il primo minore}_{k \times k} \text{ di } (a_{ij}) \text{ è } \neq 0 \}$$

$$\simeq \left\{ \left(\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \middle| \begin{matrix} b_{11} & \cdots & b_{n-k} \\ b_{k1} & \cdots & b_{k,n-k} \end{matrix} \right) \right\} \simeq \mathbb{A}^{k(n-k)}$$

Conclusione $G(k,n)$ varietà proiettiva, liscia, razionale, di dimensione $k(n-k)$.